1. A student is investigating the following statement about natural numbers.

"$n^3 - n$ is a multiple of 4"

(a) Prove, using algebra, that the statement is true for all odd numbers.
(4)

(b) Use a counterexample to show that the statement is not always true.
(1)

a)

If n is odd, then $n = 2k + 1$ (1) for an integer k.

$n^3 - n = (2k+1)^3 - (2k+1)$ (1)

$= 8k^3 + 12k^2 + 6k + 1 - (2k+1)$ (1)

$= 8k^3 + 12k^2 + 4k = 4(2k^3 + 3k^2 + k)$

As k is an integer, $4 \times (2k^3 + 3k^2 + k)$ is a multiple of 4.
Therefore, if n is odd then $n^3 - n$ is a multiple of 4. (1)

b)

Let $n = 2$.

$2^3 - 2 = 8 - 2 = 6$ which is not a multiple of 4. (1)

2. (i) A student states

"if x^2 is greater than 9 then x must be greater than 3"

Determine whether or not this statement is true, giving a reason for your answer.

(1)

(ii) Prove that for all positive integers n,

$$n^3 + 3n^2 + 2n$$

is divisible by 6

(3)

(i) claim : if $x^2 > 9$, then $x > 3$

if $x = -4$: $(-4)^2 = 16 > 9$ and $(-4) < 3$,

1

then the statement is false

$x^2 > 9$

$= x^2 - 9 > 0$

$= (x+3)(x-3) > 0$

$\therefore x > 3$, $x < -3$ (statement is false)

(ii) $n^3 + 3n^2 + 2n \equiv n(n^2 + 3n + 2)$ (1)

$= n(n+1)(n+2)$, which is the product of three consecutive integers. (1)

Examples of consecutive integers : 7, 8, 9

16, 17, 18

23, 24, 25

① one of the numbers is divisible by 3

② at least one of them will be divisible by 2

As $n(n+1)(n+2)$ is a multiple of 2 and a multiple of 3, it must be a multiple of 6. (1)

$\therefore$ So, $n^3 + 3n^2 + 2n$ is divisible by 6 for all integers n.

3. In this question p and q are positive integers with $q > p$

Statement 1: $q^3 - p^3$ is never a multiple of 5

(a) Show, by means of a counter example, that Statement 1 is **not** true.

(1)

Statement 2: When p and q are consecutive **even** integers $q^3 - p^3$ is a multiple of 8

(b) Prove, using algebra, that Statement 2 is true.

(4)

a) Let $q = 8$ and $p = 3$

$q^3 - p^3 = 8^3 - 3^3$

$= 485$ (1) which is a multiple of 5. Statement 1 is not true.

b) Let $p = 2n$ and $q = 2n+2$ (1)

$(2n+2)^3 - (2n)^3 = (2n+2)(2n+2)(2n+2) - 8n^3$

$= (4n^2 + 8n + 4)(2n+2) - 8n^3$

$= 8n^3 + 24n^2 + 24n + 8 - 8n^3$ (1)

$= 24n^2 + 24n + 8$ (1)

$= 8(3n^2 + 3n + 1)$ (1)

So, $q^3 - p^3$ is a multiple of 8.

4. (i) Use proof by exhaustion to show that for $n \in \mathbb{N}, n \leqslant 4$

$$(n+1)^3 > 3^n$$

(2)

(ii) Given that $m^3 + 5$ is odd, use proof by contradiction to show, using algebra, that m is even.

(4)

i) $n \in \mathbb{N}, n \leq 4$ so $n = 1, 2, 3$ or 4 $(0 \notin \mathbb{N})$

$n=1$, $2^3 = 8$, $3^1 = 3$ $8 > 3$

$n=2$, $3^3 = 27$, $3^2 = 9$ $27 > 9$

$n=3$, $4^3 = 64$, $3^3 = 27$ $64 > 27$

$n=4$, $5^3 = 125$, $3^4 = 81$ $125 > 81$ ①

so if $n \in \mathbb{N}, n \leq 4$, then $(n+1)^3 > 3^n$ ①

ii) Assume there exists an m which is odd such that m^3+5 is odd. ①

Set $m = 2k+1$, $k \in \mathbb{Z}$

①

$m^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 1 + 5$

$= 8k^3 + 12k^2 + 6k + 6$

$= 2(4k^3 + 6k^2 + 3k + 3)$ which is even. ①

This is a contradiction.

So if m^3+5 is odd then m must be even. ①

5. Prove, using algebra, that

$$(n + 1)^3 - n^3$$

is odd for all $n \in \mathbb{N}$

(4)

case 1: n is even
let $n = 2k$, $k \in \mathbb{Z}$

$$(2k+1)^3 - (2k)^3 = 8k^3 + 12k^2 + 6k + 1 - 8k^3 \quad (1)$$
$$= 12k^2 + 6k + 1$$
$$= 2(6k^2 + 3k) + 1$$

which is odd (1) ← because 2n is even by definition, so 2n+1 is odd

case 2: n is odd
$n = 2k+1$, $k \in \mathbb{Z}$

$$(2k+2)^3 - (2k+1)^3 = 8k^3 + 24k^2 + 24k + 8 - (8k^3 + 12k^2 + 6k + 1) \quad (1)$$
$$= 12k^2 + 18k + 7$$
$$= 2(6k^2 + 9k + 3) + 1$$

which is odd

Hence odd for all $n \in \mathbb{N}$ (1)